Some Problems on universal mappings

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Resume

First we describe inductive limits for some families of Lie algebras and groups. We also consider the linear mappings from the space $C^\infty(S', R)$ (smooth functions on a circle) to Lie algebras such that the following Local Commutatirity Property is satisfied: for ony two functions with disjoint supports the corresponding elements of the Lie algebra commute; we describe a universal mapping with this property.

1 Introduction

Mappings with the universality Property arise in many ocasions; recall, for example, that for linear spaces A, B the canonical mapping $A \times B \to A \otimes B$ is universal with respect to bilinear mappings from $A \times B$ to linear spaces. In this paper we consider some universal mappings for Lie algebras and groups. First we discuss some examples related to inductive limits of families of the Lie algebras and groups. Then (in §2) we examine linear mappings from $C^\infty(S', R)$ to Lie algebras such that the local commutativity property is fulfilled; this question originates from the "quantum field theory on a circle" ([1]).

2 Inductive limits

First recall the definition ([2]). Suppose we have a family of groups $\{G_\alpha\}$ and for any G_α and G_β a family (possibly empty) of homomorphisms $f_{\alpha\beta} : G_\alpha \to G_\beta$. A representation of the family $\{G_\alpha, f_{\alpha\beta}\}$ in a group G consists (by definition) of homomorphisms $u_\alpha : G_\alpha \to G$ such that the diagrams

$$\begin{array}{ccc} & \mathbf{G}_\alpha & \xrightarrow{u_\alpha} & \mathbf{G} \\ f_{\alpha\beta} & \downarrow & \nearrow^{u_\beta} & \\ & \mathbf{G}_\beta & & \end{array}$$

are commutative and G is generated by all $u_\alpha(G_\alpha)$. The inductive limit of our family $\{G_\alpha, f_{\alpha\beta}\}$ is, by definition, a representation $\{G^*, u^*_\alpha\}$ with the following universality Property: for any representation $\{G, u_\alpha\}$ there exists a homomorphism $p : G^* \to G$ with $u_\alpha = p \circ u^*_\alpha$ for all α.

The group G^* can be described in terms of generators and relations; but the explicite description usually turns out to be a difficult Problem.

For the family $\{L_\alpha, f_{\alpha\beta}\}$ where L_α are Lie algebras and $f_{\alpha\beta} : L_\alpha \to L_\beta$ homomorphisms the inductive limit $\{L^*, u^*_\alpha\}$ is defined in the similar way.

Now we consider some examples.

1. Heisenberg group (and Lie algebra) as inductive limit of commutative groups (commutative Lie algebras). Consider a field $K, charK \neq 2$, a K - linear space $V, dimV \geq 4$, and a bilinear form $\tau : V \times V \to K$ antisymmetrical and nondegenerate (i.e. for any $x \in V, x \neq 0$, there exists a vector $y \in V$ with $\tau(x,y) \neq 0$). A linear subspace $V_1 \subset V$ is called isotropical if τ is zero on $V_1 \times V_1$. Denote by $Is(V,\tau)$ the set of all the isotropical subspaces. Any such subspace is considered as an commutative group (with respect to the addition of vectors). For any two subspaces V_0, V_1 from $Is(V,\tau)$ such that $V_0 \subset V_1$ we have a homomorphism (inclusion) $V_0 \subset V_1$. Our goal is to describe the inductive limit of the family of groups $Is(V,\tau)$ with these homomorphisms-inclusions. To do this consider the Heisenberg group H - the set $V \times K$ with the group operation $(v_1,k_1)(v_2,k_2) = (v_1+v_2, k_1+k_2+\tau(v_1,v_2))$. For any $V_0 \in Is(V,\tau)$ we have a group homomorphism $V_0 \to H, v \mapsto (v,0), v \in V_0$.

Theorem 1 *The inductive limit of the family $Is(V,\tau)$ is the Heisenberg group H (with homomorphisms indicated above).*

Proof of Theorem 1 *Consider an arbitary representation of our family $Is(V,\tau)$ in some group G. Clearly this representation is the same thing as a mapping $\Phi : V \to G$ with the following Property:*

$$\text{if } v_i \in V, i = 1,2, \tau(v_1,v_2) = 0, \text{then } \Phi(v_1+v_2) = \Phi(v_1)\Phi(v_2), \tag{1}$$

$\Phi(0) = e$ *(the neutral element of G).*

Examine closely the mapping Φ with this Property.

Lemma 1 *We have*

$$\Phi(a+b) = \Phi(\frac{a}{2})\Phi(b)\Phi(\frac{a}{2}), \quad \forall a,b \in V, \tag{2}$$

Proof of Lemma 1 *Recall that $dimV \geq 4, charK \neq 2$. From these conditions it follows that there exist vectors $\alpha, \beta \in V$ with*

$$\tau(\alpha,a) = \tau(\alpha,b) = 0, \;\; \tau(\beta,a) = \tau(\beta,b) = 0$$

$$\tau(\alpha,\beta) + \frac{1}{4}\tau(a,b) = 0, \tag{3}$$

Write the vector $a+b$ as a sum of four vectors as follows:

$$a+b = (\frac{1}{2}a+\alpha) + (\frac{1}{2}b+\beta) + (\frac{1}{2}b-\beta) + (\frac{1}{2}a-\alpha)$$

It follows from (3) that the sum of the first and the second vectors on the right side of the last equality is orthogonal (with respect to τ) to the sum of the third and fourth vectors. Moreover the first and the second vectors are orthogonal and so are also the third and the fourth vectors. Thus the Property (1) implies

$$\Phi(a+b) = \Phi((\frac{1}{2}a+\alpha)+(\frac{1}{2}b+\beta))\Phi((\frac{1}{2}b-\beta)+(\frac{1}{2}a-\alpha)) =$$

$$= \Phi(\frac{1}{2}a+\alpha)\Phi(\frac{1}{2}b+\beta)\Phi(\frac{1}{2}b-\beta)\Phi(\frac{1}{2}a-\alpha) =$$

$$= \Phi(\frac{1}{2}a+\alpha)\Phi(b)\Phi(\frac{1}{2}a-\alpha) =$$

$$= \Phi(\frac{1}{2}a)\Phi(\alpha)\Phi(b)\Phi(-\alpha)\Phi(\frac{1}{2}a) = \Phi(\frac{1}{2}a)\Phi(b)\Phi(\frac{1}{2}a)$$

Define a mapping $w : V \times V \to G$ *by*

$$w(a,b) = \Phi(a+b)\Phi(-b)\Phi(-a), a, b \in V. \tag{4}$$

Lemma 2 *We have* $w(a,b)\Phi(c) = \Phi(c)w(a,b)$ *for all* $a, b, c \in V$.

Proof of Lemma 2 *Applying (2) repeatedly we obtain*

$$\Phi(-c)w(a,b)\Phi(c) = \Phi(-c)\Phi(a+b)\Phi(-b)\Phi(-a)\Phi(c) =$$

$$= \Phi(-a-b)(\Phi(a+b)\Phi(-c)\Phi(a+b))\Phi(-b)(\Phi(-a)\Phi(c)\Phi(-a))\Phi(a) =$$

$$= \Phi(-a-b)\Phi(2a+2b-c)\Phi(-b)\Phi(c-2a)\Phi(a) =$$

$$= \Phi(-a-b)\Phi(2a+2b-c)(\Phi(-b)\Phi(c-2a)\Phi(-b))\Phi(b)\Phi(a) =$$

$$= \Phi(-a-b)\Phi(2a+2b-c)\Phi(c-2a-2b)\Phi(b)\Phi(a) =$$

$$= \Phi(-a-b)\Phi(b)\Phi(a).$$

Thus $\Phi(-c)w(a,b)\Phi(c)$ *does not depend on* $c \in V$. *Putting* $c = 0$ *proves our Lemma.*

It follows from (1) and (4) that $w(a,b)$ *remains unchanged if we transform* (a,b) *as follows:*

$$(a,b) \to (a',b) \quad \text{where} \quad \mathrm{a} - \mathrm{a}' \perp \mathrm{a}, \mathrm{a} - \mathrm{a}' \perp \mathrm{b}, \tag{5}$$

$$\text{or} \quad (\mathrm{a},\mathrm{b}) \to (\mathrm{a},\mathrm{b}') \quad \text{where} \quad \mathrm{b} - \mathrm{b}' \perp \mathrm{a}, \mathrm{b} - \mathrm{b}' \perp \mathrm{b}, \tag{6}$$

(here $\perp$ *is orthogonalitswith respect to* τ*). Clearly* $\tau(a,b)$ *also remains intact under transformation (5),(6).*

Lemma 3 *Let* $a, b, a', b' \in V$ *and* $\tau(a,b) = \tau(a',b') \neq 0$. *Then the pair* (a',b') *can be obtained from* (a,b) *applying a sequence of transformations of the form (5) and (6).*

Proof of Lemma 3 *Let* $V(a,b)$ *and* $V(a',b')$ *be linear subspaces spanned by* a, b *and* a', b'. *Let* n *be a rank of the bilinear form* $\tau(x,y), x \in V(a,b), y \in V(a',b')$; *notice that* n *is simply the rank of the matrix*

$$\begin{pmatrix} \tau(a,a') & \tau(a,b') \\ \tau(b,a') & \tau(b,b') \end{pmatrix}$$

Consider the following cases. Case 1; $n = 0$. *Thus* $V(a,b)$ *is orthogonal to* $V(a',b')$. *In this case the Lemma follows if we consider the transformations* $(a,b) \to (a+a',b) \to (a+a',b') \to (a',b')$. *Case 2;* $V(a,b) = V(a',b')$. *Take vectors* a, b *orthogonal to* $V(a,b)$ *and* $\tau(a_1,b_1) = \tau(a,b)$. *Applying the case 1 we can pass from* (a,b) *to* (a_1,b_1) *(applying a sequence of transformations of the form (5), (6)) and then pass from* (a_1,b_1) *to* (a',b'). *Case 3;* $n = 1$. *Applying the case 2 we can assume that* $a' \perp a, a' \perp b, b' \perp b, \tau(a,b') = \tau(a,b)$. *The desired transformations are* $(a,b) \to (a,b') \to (a',b')$. *Case 4;* $n = 2$. *Applying the case 2 we can assume that* $a' \perp a, b' \perp b, \tau(a',b) = \tau(a,b)$. *Then to prove our Lemma use the transformations* $(a,b) \to (a',b) \to (a',b')$.

Now we can conclude the Proof of the Theorem 1 as follows. From Lemma 3 it follows that if $\tau(a,b) = \tau(a',b')$ then $w(a,b) = w(a',b')$. Thus $w(a,b) = r(\tau(a,b))$ for some mapping $r : K \to G$. From (4) and Lemma 2 it follows that $w(a+b,c)w(a,b) = w(a,b+c)w(b,c)$, $w(a,0) = e$. This easily gives $r(0) = e, r(x+y) = r(x)r(y)$ for any $x, y \in K$. It follows that the mapping $H \to G, (v,k) \mapsto \Phi(v)r(k)$ is a homomorphism. This clearly proves our Theorem.

Now consider subspaces $V_0 \in Is(v,\tau)$ as commutative Lie algebras (over the field K). The corresponding inductive limit is the Heisenberg Lie algebra $V \otimes K$ with the Lie bracket $[(v_1,k_1),(v_2,k_2)] = (0,\tau(k_1,k_2))$. We do not dwell on this in details.

2. Lie algebras of vector fields. Consider two examples.

a) Let (X,w^2) be a compact connected symplectic manifold; (see, for example, the book [3], pp.123 for all the notions used in this section). By $V(X,w^2)$ denote the Lie algebra of vector fields preserving the form w^2 and by $V_0(X,w^2)$ the Lie subalgebra hamiltonian vector fields. So, for any $\xi \in V_0(X,w^2)$ we have a function (hamiltonian) $f_\xi \in C^\infty(X,w^2)$ such that $df_\xi = i(\xi)w^2$ where $i(\xi)$ denotes an inner product; f_ξ is defined up to an additive constant.

Consider all the domains $Y \subset X$ diffeomorphic to $R^{2n}, dimX = 2n$ (we write $Y \simeq R^{2n}$). For any such Y denote by $V_o(Y,w^2)$ the Lie subalgebra of all vector fields $\xi \in V(X,w^2)$ supported in Y (i.e. ξ is zero outside of a compact subset of Y). If $Y_1 \subset Y_2$ then we have an obvious inclusion homomorphism $V_0(Y_1,w^2) \to V_0(Y_2,w^2)$. Now we describe the inductive limit of the family of Lie algebras $\{V_0(Y,w^2), Y \simeq R^{2n}\}$.

Consider the direct sum of Lie algebras $V_0(X,w^2) \oplus R$ and for any $Y \subset X, Y \simeq R^{2n}$, define an inclusion $V_0(Y,w^2) \to V_0(X,w^2) \oplus R, \xi \mapsto (\xi, \int_X f_\xi(x)w^{2n})$ where f_ξ is the hamiltonian of ξ supported in Y.

Theorem 2 *The Lie algebra $V_0(X,w^2) \oplus R$ with the inclusions indicated above is the inductive limit of $\{V_0(Y,w^2), Y \subset X, Y \simeq R^{2n}\}$.*

The similar result for the corresponding family of diffeomorphism groups is also valid: we do not consider this case.

b) Let (X,v^n) be a compact connected manifold, $dimX = n, n \geq 3$, equipped with a volume form v^n (so X is oriented). For any domain $Y \subset X, Y \simeq R^n$ denote $V_0(Y,v^n)$ the Lie algebra of vector fields preserving the form v^n and supported in Y. By $V_0(X,v^n)$ denote the linear span of all $\{V_0(Y,v^n), Y \subset X, Y \simeq R^n\}$. Now we describe the inductive limit of the family of Lie algebras $\{V_0(Y,v^n), Y \subset X, Y \simeq R^n\}$ with obvious inclusion homomorphisms $V_0(Y_1,v^n) \subset V_0(Y_2,v^n)$ for $Y_1 \subset Y_2$.

Let $E^k = E^k(X)$ be the space of exterior k-forms on$X$, $Z^k = Z^k(X)$ and $B^k = B^k(X)$ the subspaces of closed and exact forms; put $H^k = H^k(X) = Z^k/B^k$ (k- domensional cohomologies of X). Consider the linear space E^{n-2}/B^{n-2} and introduce a bracket $[\ ,\ ]$ in it as follows. If θ_1 and θ_2 are $(n-2)$-forms supported in a domain $Y \subset X, Y \simeq R^n$, then put $[\theta_1 + B^{n-2}, \theta_2 + B^{n-2}] = \theta_3 + B^{n-2}$ where $\theta_3 \subset Y$ and vector fields $\xi_k \in V_0(Y,v^n), k = 1,2,3$, defined by $i(\xi_k)v^n = d\theta_k$ satisfy the equality $[\xi_1,\xi_2] = \xi_3$. The bracket $[\ ,\ ]$ is uniqly determined by this rule and makes E^{n-2}/B^{n-2} a Lie algebra. Moreover it is a central extension of $V_0(X,v^n)$ by the centre $H^{n-2}(X)$ considered as acommutative Lie algebra. For any $Y \subset X, Y \simeq R^n$ we have an inclusion $V_0(Y,v^n) \to E^{n-2}/B^{n-2}, \xi \mapsto \theta + B^{n-2}$, where $i(\xi)v^n = d\theta$ and $supp\theta \subset Y$.

Theorem 3 *The Lie algebra E^{n-2}/B^{n-2} is the inductive limit of the family of Lie algebras $\{V_0(Y,v^n)Y \subset X, Y \simeq R^n\}$.*

Similar results for diffeomorphism groups were obtained in [4]; they are more complicated to formulate (and to prove).

3 Locally commutative mappings from the space $C^\infty(S^1,R)$ to Lie algebras

Let $C^\infty(S^1,R)$ be the spase of smooth real functions on the circle S^1 and L an arbitrary real Lie algebra. A linear mapping $f : C^\infty(S^1,R) \to L$ is called locally commutative if for any two functions u,v from $C^\infty(S^1,R)$ having disjoint supports we have $[f(u),f(v)] = 0$. Consider all the pairs (L,f) where L is aa Lie algebra and f a locally commutative mapping from $C^\infty(S^1,R)$ to L. The pair (L^*,f^*) is calleduniversalif for any pair (L,f) there exists a Lie algebra homomorphism $p : L^* \to L$ with $f = p \circ f^*$ and $f^*(L^*)$ generates L^* as a Lie algebra. Now we describe (L^*,f^*).

Consider the associative algebra $A = A_1 \oplus A_2 \oplus \ldots$ where $A_1 = C^\infty(S^1,R), A_k$ is the space of jets of C^∞-functions $u(x_1,\ldots,x_n), x_i \in S^1$ in the neighbourhood of the "diagonal" $x_1 = \ldots = x_n$. If $u \in A_m, v \in A_n$ then $uv \in A_{m+n}$ is defined as a jet of the function $u(x_1,\ldots,x_m)v(x_{m+1},\ldots,x_{m+n})$. Consider A as a Lie algebra with $[u,v] = uv - vu$ and denote by L^* the Lie subalgebra generated by A_1. We have a mapping $f^* : C^\infty(S^1,R) = A_1 \subset L^*$.

Theorem 4 *The pair (L^*,f^*) is universal.*

To conclude this section slightly modify the notion of a locally commutative mapping. Consider now only topological (locally convex) Lie algebras L and locally commutative mappings $f : C^\infty(S^1,R) \to L$ subjected to the continuity condition which we now formulate. The mapping f has a prolongation $\tilde{f}$ to the free Lie algebra $F = F_1 \oplus F_2 \oplus \ldots$ over the linear space $C^\infty(S^1,R)$. Any F_k is embedded into $(C^\infty(S^1,R))^{\otimes k} \subset C^\infty((S^1)^k,R)$; this last space consists of smooth functions $u(x_1,\ldots,x_n), x_i \in S^1$. Consider only the mappings f for which the mapping $\tilde{f}$ is continious on any F_k with respect to Sobolev norm $\|\cdot\|_p$ (maximum of modules of derivativer of order $\leq p$) restricted to $F_k \subset C^\infty((S^1)^k,R)$. The corresponding universal pair (for fixed p) denote by (L^*_p,f^*_p). The explicite description of it can be deduced from Theorem 4. We only consider the case $p = 1$. It turns out the L^*_1 can be realized as the space of polinomials $u_1T + \ldots + u_mT^m$ where $u_i \in C^\infty(S^1,R)$ and T is a formal variable. The Lie bracket is defined by $[u_kT^k, v_mT^m] = 0$ if $k \geq 2, m \geq 2, [u_1T, v_mT^m] = u_1v_mT^{m+1}$, if $m \geq 2$ and $[u_1T, v_1T] = (u_1v_1' - u_1'v_1)T^2$.

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